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THE 2-PRIMARY K-THEORY ADAMS SPECTRAL SEQUENCE

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1. Introduction

The main purpose of this article is to give an elementary account of the bigraded ring structure of the E_2-term of the Adams spectral sequence for the sphere spectrum defined by complex K-theory localized at the prime two. As a consequence of the particularly simple structure of this bigraded ring, it is possible, with the aid of convergence, to determine the differentials and the limit group of this spectral sequence. It is therefore a subsidiary concern of the article to catalogue the basic facts involved in calculating with the sequence.

In order to state our main result let us set some notation and terminology. Let K denote the BU-spectrum localized at the prime two. It is well-known that this E_2-term is the cohomology of $K_*(K)$: for all s and t ($s \geq 0$)

$$E_2^{s,t} = H^{s,t}(K_*(K)) = \mathrm{Ext}^{s,t}_{K_*(K)}(K_*(S^0), K_*(S^0)).$$

Adams, Harris, and Switzer [4] have shown that the $Z_{(2)}[t, t^{-1}]$-bimodule $K_*(K)$ is a certain subring of $Q[u, v, u^{-1}, v^{-1}]$. In particular, they located elements

$$g_n = (v^n - u^n)/2^{d(n)} \in K_{2n}(K),$$

where $d(n)$ is the highest power of 2 dividing $3^{|n|} - 1$ and n is a non-zero integer. Finally let $C = K_*(K)$ and if g is a generator of the cyclic $Z_{(2)}$-module M, let us denote this relationship by $M\langle g \rangle$.

The bigraded ring structure of the cohomology of $C, H^{**}(C)$, is given in

Theorem 1.1.

(a) $$H^{0,n}(C) = \begin{cases} Z_{(2)}\langle 1 \rangle, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

(b) $$H^{1,n}(C) = \begin{cases} (Z/2^{d(t)})\langle g_t \rangle, & n = 2t \neq 0, \\ 0, & n = 0 \text{ or } n \text{ odd}. \end{cases}$$

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Here we have denoted the cohomology class of g_t by g_t.

(c)
$$H^{2,n}(C)=\begin{cases} Q/Z_{(2)}\oplus(Z/2)\langle g_1g_{-1}\rangle, & n=0,\\ (Z/2)\langle g_{-1}g_2\rangle, & n=2,\\ (Z/2)\langle g_1g_{t-1}\rangle, & n=2t\neq 0,2,\\ 0, & \textit{otherwise}.\end{cases}$$

In $H^{2,0}(C)$ the $Q/Z_{(2)}$-summand is spanned by the elements $g_{2^i}g_{-2^i}$ with $i>0$ and these elements are infinitely 2-divisible.

(d) *For $s>2$,*

$$H^{s,n}(C)=\begin{cases} (Z/2)\langle g_1^{s-2}g_{-1}g_2\rangle, & n=2s-2,\\ (Z/2)\langle g_1^{s-1}g_{t-s+1}\rangle, & n=2t \textit{ but } \neq 2s-2,\\ 0, & \textit{otherwise}.\end{cases}$$

(e) *The relations*

$$g_{-m\cdot 2^i}g_{m\cdot 2^i}=g_{-2^i}g_{2^i} \quad \textit{with } m \textit{ odd}, \ i\geq 0,$$

$$g_{2r}g_{2t}=0 \quad \textit{with } r,t\neq 0, t\neq -r,$$

$$g_{2r+1}g_n=g_1g_{2r+n} \quad \textit{with } n\neq 0, -2r,$$

$$g_{2r+1}g_{-2r}=g_{-1}g_2 \quad \textit{with } r\neq 0,$$

$$g_{-2^i}g_{2^i}=2g_{-2^{i+1}}g_{2^{i+1}}+2^{3+i}g_{-2^{i+2}}g_{2^{i+2}}$$

with $i>0$ and those implied by these are exactly the relations among products in $H^{**}(C)$.

One can summarize Theorem 1.1 by saying that $H^*(C)$ with $*\geq 3$ is the $(Z/2)[g_1]$-module generated by the generators of H^0, H^1 and the products $g_{-1}g_2, g_{-2^i}g_{2^i}$, $i>0$, subject to the relations $g_1^m\cdot g_{-2^i}g_{2^i}=0$ when i, $m>0$.

Using the identification $K_*(K)\subset Q[u,v,u^{-1},v^{-1}]$ (see also 2.1 below) we recall the structure maps of $C=K_*(K)$ in the next formulae:

(a) the units $\eta_L, \eta_R : R=Z_{(2)}[t,t^{-1}]\to C$ are given by $\eta_L(t)=u$, $\eta_R(t)=v$;

(b) the augmentation $\varepsilon : C\to R$ is given by $\varepsilon(f(u,v))=$ f(t;t);

(c) the product $\mu : C\otimes C\to C$ is multiplication of polynomials;

(d) the coproduct (or diagonal) $\Psi : C\to C\otimes_R C$ is given by $\Psi(u)=u\otimes 1$ and $\Psi(v)=1\otimes v$.

The cohomology of $C=K_*(K)$ cannot be computed by the usual methods (e.g. the May spectral sequence, the Adams' change of rings spectral sequence for Hopf algebras, etc.) since the left and right actions of $Z_{(2)}[t,t^{-1}]$ on $K_*(K)$ do not agree; instead Theorem 1.1 is deduced from the exact sequence

$$\cdots\to H^{s,t}(C)\xrightarrow{2_*} H^{2,t}(C)\xrightarrow{j_t} H^{s,t}(C')\xrightarrow{\delta} H^{s+1,t}(C)\to\cdots$$

where C' is the Hopf algebra obtained from C by tensoring all of its structure maps with $Z/2$. There result then commutative diagrams

$$\begin{array}{ccccccc} R & \xrightarrow{\eta_L,\eta_R} & C & \quad & C & \xrightarrow{\varepsilon} & R \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ R' & \xrightarrow[\eta'_L,\eta'_R]{} & C' & & C' & \xrightarrow[\varepsilon']{} & R' \end{array}$$

$$\begin{array}{ccccccc} C\otimes C & \xrightarrow{\mu} & C & \quad & C & \xrightarrow{\Psi} & C\otimes_R C \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j\otimes j} \\ C'\otimes C' & \xrightarrow[\mu']{} & C' & & C' & \xrightarrow[\Delta]{} & C'\otimes_{R'} C' \end{array}$$

where $\Delta = \Psi\otimes Z/2$, $\eta'_L = \eta'_L\otimes Z/2$, etc. and j denotes the map obtained by tensoring with $Z_{(2)}\to Z/2$. Once sufficient information about the Hopf algebra structure of C' is known, its cohomology can be computed by means of Adams' changes of rings spectral sequence for Hopf algebras [1]. The Hopf algebra structure of C' is given in

Theorem 1.2. (a) *The left and right action of* $R' = (Z/2)[t, t^{-1}]$ *on* C' *coincide.*

(b) C' *is the polynomial algebra over* R' *on generators* $\{w_i\}_{i\geq 0}$ *of degree* 0 *subject to the relations* $w_0 = 1$ *and for* $i > 0$, $w_i^2 = w_i$.

(c) *The diagonal* Δ *of* C' *satisfies*

$$\Delta(w_{k+1}) = w_{k+1}\otimes 1 + 1\otimes w_{k+1} \quad \textit{for } k = 0, 1$$

and for $k \geq 2$

$$\Delta(w_{k+1}) = w_{k+1}\otimes 1 + 1\otimes w_{k+1} + w_k\otimes w_k + \sum_s m'_s\otimes m''_s,$$

where m'_s *and* m''_s *are monomials in* $w_1, \ldots, w_k$, *at least one which contains a factor* w_i *with* $i < k$. *In particular,*

$$\Delta w_3 = w_3\otimes 1 + 1\otimes w_3 + w_2\otimes w_2 + w_2\otimes w_1 + w_1\otimes w_2 + w_1\otimes w_1.$$

Let x and y in $H^{1,0}(C')$ be the cohomology classes of w_1 and $w_1 + w_2$ respectively. Then the cohomology $H^{**}(C')$ is given in

Theorem 1.3. $H^{**}(C') = P_{R'}[x]\otimes_{R'} E_{R'}(y)$ *where* $P_{R'}$ *and* $E_{R'}$ *stand for polynomial and exterior algebra respectively.*

As an application of Theorem 1.1 we will determine the differentials in the

K-theory spectral sequence for S^0 and in terms of them determine its limit group $\pi_*^K(S^0)$, the K-completion of the homotopy groups of the sphere spectrum $\pi_*(S^0)$. As a general rule in order to get geometric information of an Adams spectral sequence, some geometric 'prime' seems to be required. In the present case this 'prime' will take the form of assumptions on $\pi_{-1}^K(S^0)$ and $\pi_2^K(S^0)$. The assumptions which we are going to make are certainly true and while our arguments might be more satisfying (or at least more complete) if we were to supply verifications for these statements instead of simply assuming them, this would take us too far from our purpose (as these verifications unavoidably involve a considerable amount of localization theory applied to the BO spectrum for real K-theory).

So for our next result, in addition to Theorem 1.1 and the convergence of the spectral sequence [5], we will assume known that

$$\pi_{-1}^K(S^0)=\pi_{-1}(S^0)=0 \quad \text{and} \quad \pi_2^K(S^0)=\pi_2(S^0)=Z/2.$$

Theorem 1.4. (a) *The following elements survive to* E_∞^{**}:

$$1\in E_2^{0,0}$$

$$g_n\in E_2^{1,2n} \quad \textit{for } n=4i\neq 0,\ n=4i+1, \quad \textit{and}$$

$$g_{-1}g_2\in E_2^{2,2}.$$

(b) *For* $n=4i+2$ *or* $4i+3$, $d_3(g_n)=g_1^3g_{n-2}$.

(c) *The values of the differentials on the remaining elements are given by Theorem 1.1 and the Leibniz formula.*

The next result which gives the values of $\pi_*^K(S^0)$ is the immediate corollary to Theorem 1.4 which one would expect.

Corollary 1.5. *The groups* $\pi_i^K(S^0)$ *are:*

$Z_{(2)}\oplus Z/2$ *for* $i=0$,

$Q/Z_{(2)}$ *for* $i=-2$,

$Z/2$ *for* $i\equiv 0 \pmod 8$, *with* $i\neq 0$,

$Z/2\oplus Z/2$ *for* $i\equiv 1 \pmod 8$,

$Z/2$ *for* $i\equiv 2 \pmod 8$,

$Z/8$ *for* $i\equiv 3 \pmod 8$,

$Z/2^{d((i+1)/2)}$ *for* $i\equiv 7 \pmod 8$, *with* $i\neq -1$,

and 0 *otherwise.*

The proofs are organized as follows. In Section 2 Theorem 1.1 is deduced from Theorem 1.3 and several technical lemmas. In Section 3 Theorems 1.2 and 1.3 are proved. In the final section the results of the previous sections are applied to obtain Theorem 1.4 and Corollary 1.5.

Finally a few words about priority. The author is informed by Andrew Baker that D.C. Ravenel knows both the E_2-term and the differentials of the K-theory spectral sequence for S^0; the author supposes J.F. Adams does also. Moreover Adams, Baird and Ravenel [5] have a determination of $\pi_*^K(S^0)$ obtained by different methods.

2. Proof of Theorem 1.1

Before beginning the proof certain technical matters need to be disposed of. As the cohomology of $K_*(K)$ is the homology of the cobar complex, it will be convenient to set notation for it by recalling its definition. If C is a Hopf ring of co-operations over R (among other things we are assuming C is a bimodule over R with augmentation $\varepsilon : C \to R$; see [10, p. 415] for definition), then the cobar construction is the cochain complex whose sth term $\Omega^s(C)$ is the s-fold tensor product of C and whose differential

$$d^s : \Omega^s(C) \to \Omega^{s+1}(C)$$

is given by the formula

$$d^s(\alpha_1 \otimes \cdots \otimes \alpha_s) = 1 \otimes \alpha_1 \cdots \otimes \alpha_s + (-1)^{s+1} \alpha_1 \otimes \cdots \otimes \alpha_s \otimes 1 + \sum_{i=1}^{s} (-1)^i \alpha_1 \otimes \cdots \otimes \Psi \alpha_i \otimes \cdots \otimes \alpha_s$$

where $\Psi : C \to C \otimes_R C$ is the co-product of C. The reduced co-bar complex $\bar{\Omega}^*(C)$ is the sub-complex of $\Omega^*(C)$ defined by $\bar{\Omega}^s(C) = \otimes^s \ker \varepsilon$. If the left and right actions of R on C coincide, then the definition of $\bar{\Omega}^*(C)$ given above agrees with the usual one [1].

In order to choose suitable cocycle representatives for the generators of $H^{**}(K_*(K))$ and to establish relations among products of cohomology clases it will be convenient to recall the main result of Adams, Harris and Switzer in [4] as adapted to the 2-local case. As $\pi_* K = Z_{(2)}[t, t^{-1}]$ and $\pi_* K \otimes \pi_* K \otimes Q \cong K_*(K) \otimes Q$, there is a monomorphism

$$K_*(K) = K_*(K) \otimes Z_{(2)} \to K_*(K) \otimes Q = Q[u, v, u^{-1}, v^{-1}].$$

Adams, Harris and Switzer characterized its image in the following

Theorem. *A Laurent polynomial* $f(u, v) \in Q[u, v, u^{-1} v^{-1}]$ *lies in* $K_*(K)$ *if and only if*

$$f(t, (2r+1)t) \in Z_{(2)}[t, t^{-1}] \quad \textit{for all } r \in Z. \tag{2.1}$$

Condition (2.1) is called the *integrality condition* and polynomials which satisfy it are said to be *integral.*

The proof proceeds in three steps.

(a) If f is in $K_*(K)$, f is shown to satisfy (2.1) by means of Adams' operations $\psi^{2r+1}: K \to K$ (which are defined since K is 2-local).

(b) The elements

$$p_n'(u,v) = \frac{v(v-u)\cdots(v-(n-1)u)}{n!}$$

clearly satisfy (2.1) as do the binomial coefficients

$$\binom{w}{n} = \frac{w(w-1)\cdots(w-n+1)}{n!}$$

where $w = vu^{-1}$ so that

$$p_n'(u,v) = u^n \binom{w}{n}.$$

An inductive argument using the binomial coefficients shows that if f satisfies (2.1), then it is a $Z_{(2)}[u, v, u^{-1}, v^{-1}]$-linear combination of the p_n'.

(c) Finally, the p_n' are shown to lie in $K_*(K)$ by analyzing the effect of the Bott map in the K-homology direct limit which defines $K_*(K)$.

We remark in passing that the polynomials g_n defined earlier satisfy (2.1) for reasons of number theory worked out by Adams in $J(X)$–II.

To prepare the way for the study of relations, we shall also need the following notation.

For $y = w \otimes z \in [\Omega^2(K_*(K))]_{2n}$ with $w = \sum_{i+j=n} a_{ij} u^i v^j$, set $\zeta(y) = a_{0n} u^n z$; for $y = \sum_\alpha y_\alpha \in \Omega^2(K_*(K))$ with $y_\alpha = w_\alpha \otimes z_\alpha$ define $\zeta(y) = \sum_\alpha \zeta(y_\alpha)$. In terms of this notation, we have the following

Lemma 2.2. *For* $y = \sum_\alpha w_\alpha \otimes z_\alpha \in [\Omega^2(K_*(K))]_{2n}$ *with* $d^2 y = 0$, *set*

$$\zeta(y, a) = \zeta(y) + a(v^n - u^n)$$

with $a \in Q$. *Then* $y \in \operatorname{im} d^1$ *if and only if for some* a *in* Q

$$\zeta(y, a)(t, (\pm 3t)) \in Z_{(2)}[t, t^{-1}].$$

The proof of 2.2 is sketched at the end of Section 3.

With these preliminaries out of the way we can now begin the proof of Theorem 1.1.

Proposition 2.3. *In* $H^{**}(K_*(K))$

$$g_{-m\cdot 2^i} g_{m\cdot 2^i} = g_{-2^i} g_{2^i}, \quad m \text{ odd},\ i > 0,$$

$$g_{2r} g_{2t} = 0, \quad r, t \neq 0,\ t \neq -r,$$

$$g_{2r+1} g_n = g_1 g_{2r+n}, \quad n \neq 0,\ -2r$$

$$g_{2r+1} g_{-2r} = g_{-1} g_2, \quad r \neq 0.$$

Further the groups $H^{s,*}(K_*(K))$ are torsion for $s \geq 1$ and in particular the elements $g_1 g_n$ with $n \neq 0$ and $g_{-1}g_2$ have order at most 2 as do their g_1^m multiples. Finally for $i>0$ the elements $g_{-2^i}g_{2^i}$ have order 2^{i+2}, are infinitely 2-divisible and

$$g_{-2^i}g_{2^i} = 2g_{-2^{i+1}}g_{2^{i+1}} + 2^{3+i}g_{-2^{i+2}}g_{2^{i+2}}.$$

Proof. All assertions about elements in $H^{2,*}(C)$ $(C = K_*(K))$ are immediate consequences of 2.2. For example if

$$y = g_{-2^i} \otimes g_{2^i} - 2g_{-2^{i+1}} \otimes g_{2^{i+1}} - 2^{3+i}g_{-2^{i+2}} \otimes g_{2^{i+2}}$$

$$\zeta(y) = \frac{1}{2^{i+2}}u^{-2^i}g_{2^i} - \frac{1}{2^{i+2}}u^{-2^{i+1}}g_{2^{i+1}} - \tfrac{1}{2}u^{-2^{i+2}}g_{2^{i+2}}$$

is integral since $\zeta(y)(1,-3) = \zeta(y)(1,3)$ is in $Z_{(2)}$. The claim about order of $g_{-2^i}g_{2^i}$ is established as follows. For $y_i = g_{-2^i} \otimes g_{2^i}$, $\zeta(2^{i+2}y_i)$ is integral so $2^{i+2}y_i$ bounds. On the other hand, there is no number $a \in Q$ so that

$$\zeta(4y_1, a)(t, (\pm 3t)) \in Z_{(2)}[t, t^{-1}].$$

Hence $4g_{-2}g_2 \neq 0$ in $H^{2,0}(C)$ while $2^{i+1}g_{-2^i}g_{2^i} = 4g_{-2}g_2$.

The values of $H^{1,*}(C)$ are well-known [4, 10]. To show that $H^{m,*}(C)$ is torsion for $m \geq 2$, we employ a generalization of the element $\zeta(y)$ of 2.2: for $y = \sum_\alpha w_\alpha \otimes z_\alpha$ with $z_\alpha \in \Omega^{m-1}(C)$ and $w_\alpha = \sum_{i+j=n_\alpha} a_{ij}^\alpha u^i v^j$ set $\zeta(y) = \sum_\alpha a_{0n_\alpha} u^{n_\alpha} z_\alpha$. Then if $d^m y = 0$, $d^{m-1}(\zeta(y)) = y$. In general $\zeta(y)$ is only an element of $\otimes^{m-1} Q[u, v, u^{-1}, v^{-1}]$ and not of $\Omega^{m-1}(C)$. However for some power of 2, say 2^n, $2^n\zeta(y) \in \Omega^{m-1}(C)$ and then $d^{m-1}(2^n\zeta(y)) = 2^n y$. Similarly using the construction ζ one can show that g_1^m multiplies $(m \geq 0)$ of $g_1 g_n$ and $g_{-1}g_2$ have order at most 2 in $H^{**}(C)$. This completes the proof.

We are now in a position to obtain the coboundary $\delta : H^{*,*}(C') \to H^{*+1,*}(C)$ where $C = K_*(K)$ and $C' = K_*(K)/2$.

Proposition 2.4. *The value of $\delta^m : H^{m,2n}(C') \to H^{m+1,2n}(C)$ for $m>0$ is given by the formulae:*

(a) $$\delta^m(t^n x^m) = \left\{ \left(\overset{m}{\otimes} g_1 \right) \otimes (v^{n-m} - u^{n-m}) \Big/ 2 \right\},$$

(b) $$\delta^m(t^m x^{m-1} y) = \left\{ \left(\overset{m-1}{\otimes} g_1 \right) \otimes g_2 \otimes (v^{n-m-1} - u^{n-m-1}) \Big/ 2 \right\}.$$

Proof. Formula (a) follows from the fact that $u^{n-m}g_1 \otimes (\otimes^{m-1} g_1) \in \Omega^m(C)$ projects to cocycle in $\Omega^m(C')$ which represents $t^n x^m$, while formula (b) follows from the fact that $\otimes^{m-1} g_1 \otimes u^{n-m-1} g_2 \in \Omega^m(C)$ projects to a cocycle in $\Omega^m(C')$ which represents $t^n x^{m-1} y$ since $j^{n-m-1} g_2 = u^{n-m-1}(v^2 - u^2)/8$ projects to $t^{n-m+1}(w_1 + w_2)$ in $K_*(K)/2$.

Corollary 2.5. *For* $m>0$,

(a) $$\delta^m(t^n x^m)=\begin{cases}0, & n-m \text{ is even},\\ g_1^m g_{n-m} & n-m \text{ is odd;}\end{cases}$$

(b) $$\delta^m(t^n x^{m-1}y)=\begin{cases}0, & n-m \text{ is odd but } \neq -1,\\ 0, & n-m=-1 \text{ and } m>1,\\ 4g_2 g_{-2}, & n-m=-1,\ m=1,\\ g_1^m g_{n-m}, & n-m \text{ is even but } \neq 0,\\ g_1^{m-1} g_2 g_{-1}, & n-m=0.\end{cases}$$

The proof is immediate in view of 2.3 and 2.4 and the definition of the g_i's.

The next step is to define a certain subring of $H^{**}(C)$. let $B^{s,t}$ be the subgroup of $H^{s,t}(C)$ asserted to be the value of $H^{s,t}(C)$ in Theorem 1.1. For example, $B^{2,2}=(Z/2)\langle g_{-1}g_2\rangle \subset H^{2,2}(C)$. It is our plan to show that $B^{s,t}=H^{s,t}(C)$ for all s,t. This is well-known for $s=0,1$. For the present we note that $\delta^m: H^{m,n}(C')\to H^{m+1,n}(C)$ actually takes its values in $B^{m+1,n}$ and that B^{**} is a bi-graded subring of $H^{**}(C)$.

Theorem 2.6. *The sequence*

$$\cdots \to B^{s,t} \xrightarrow{2_*} B^{s,t} \xrightarrow{j_*} H^{s,t}(C') \xrightarrow{\delta^s} B^{s+1,t} \to \cdots$$

is exact, where 2_* *and* j_* *are the restrictions to* $B^{*,*}$ *of the maps induced by* $2: C\to C$ *and* $j: C\to C'=C/2C$.

Proof. Since 2_* is just multiplication by 2, the theorem will follow from 2.3 and 2.5 once the behavior of j_* is known. Since j_* is a ring homomorphism, its behavior on $B^{s,t}$ with $s>1$ is determined by Theorem 1.3 and its behavior on $B^{1,*}$ which is given by the formula

$$j_*(g_n)=\begin{cases}t^n x, & \text{if } n \text{ is odd},\\ t^n y, & \text{if } n \text{ is even but } \neq 0.\end{cases}$$

This follows easily from (2.1); for if n is odd, then

$$f(u,v)=\tfrac{1}{2}[(v^n-u^n)/2-u^{n-2}p_2'(u,v)]$$

is integral while if $n\neq 0$ is even

$$g(u,v)=\tfrac{1}{2}[v^n-u^n)/2^{d(n)}-u^{n-2}p_2'(u,v)-u^{n-4}p_4'(u,v)]$$

is integral. As a consequence

$$j_*(g_1^m g_n)=\begin{cases}t^{m+n}x^{m+1}, & n \text{ odd},\\ t^{m+n}x^m y, & n \text{ even } (\neq 0)\end{cases}$$

and

$$j_*(g_1^m g_2 g_{-1})=t^{m+1}x^{m+1}y$$

while

$$j_*(g_{-2^i}g_{2^i})=y^2=0, \quad \text{if } i>0.$$

It follows therefore that the generators in Theorem 1.1 have the orders claimed and that the sequence of the present theorem is exact.

Modulo the proofs assigned to the next section, the proof of Theorem 1.1 is completed by the next proposition.

Proposition 2.7. *In the exact commutative ladder*

$$\begin{array}{ccccccccccc} 0 \longrightarrow \operatorname{im}\delta^1 & \longrightarrow & B^{2,*} & \xrightarrow{2_*} & B^{2,*} & \xrightarrow{j_*} & H^{2,*}(C') & \xrightarrow{\delta^2} & B^{3,*} & \xrightarrow{2_*} & \cdots \\ \| & & \downarrow g^{2,*} & & \downarrow g^{2,*} & & \| & & \downarrow g^{3,*} & & \\ 0 \longrightarrow \operatorname{im}\delta^1 & \longrightarrow & H^{2,*}(C) & \xrightarrow{2_*} & H^{2,*}(C) & \xrightarrow{j_*} & H^{2,*}(C') & \xrightarrow{\delta^2} & H^{3,*}(C) & \xrightarrow{2_*} & \cdots \end{array}$$

the inclusions labelled $g^{s,*}$ with $s \geq 2$ are in fact identity maps.

Proof. In an inductive manner apply the following fact about torsion groups: if in the exact commutative ladder

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1 & \longrightarrow & B & \longrightarrow & B & \longrightarrow & H^2 \\ & & \downarrow f^1 & & \downarrow g & & \downarrow g & & \downarrow f^2 \\ 0 & \longrightarrow & K^1 & \longrightarrow & H & \longrightarrow & H & \longrightarrow & K^2 \end{array}$$

B, H are 2-torsion and f^1, f^2 are isomorphisms, then g is an isomorphism. (A version of this involving torsion modules over more general rings appears in [8]).

3. Proof of Theorems 1.2 and 1.3

We begin by deducing 1.3 from 1.2 by means of the Adams' change of rings spectral sequence for Hopf algebras [1]. As before let $C' = K_*(K)/2$. We may filter C' by sub Hopf algebras C'_k: C'_k is the sub Hopf algebra generated by $w_1, \ldots, w_k$. For each $k \geq 1$

$$R' \to C'_k \xrightarrow{i} C'_{k+1} \xrightarrow{\pi} C'_{k+1}//C'_k \to R'$$

is an injective extension sequence [9] ($C'_{k+1}//C'_k = C'_{k+1}/I(C'_k) \cdot C'_{k+1}$, where $I(C'_k)$ is the augmentation ideal.) Thus for each $k \geq 1$, there is a multiplicative spectral sequence $\xi(k) = \{E_r(k), d_r(k)\}$ for which

$$E_2^{p,q}(k) = H^p(C'_k) \otimes H^q(C'_{k+1}//C'_k) \Rightarrow H^{p+q}(C'_{k+1}).$$

The spectral sequence is constructed by filtering the reduced cobar complex $\bar{\Omega}^*(C'_{k+1})$ as follows:

$$f_1 \otimes \cdots \otimes f_s \in F^p \bar{\Omega}^s(C'_{k+1})$$

if $\pi(f_i)=0$ for p values of i.

The formula for Δw_k in C' can be interpreted readily in the spectral sequence $\xi(k)$.

Lemma 3.1. *Let $z_k = \{d(w_{k+1})\} \in H^2(C'_k)$ where d is the cobar differential and $\{\cdot\}$ denotes homology class.*

(a) $$d_2^{0,1}\{w_{k+1}\} = z_k = \begin{cases} 0 & \text{for } k=0,1 \\ \neq 0 & \text{for } k>1 \end{cases}$$

in $E_2^{2,0}(k) = H^2(C'_k)$.

(b) *For $k>0$, the projection*

$$\bar{\Omega}^2(C'_{k+1}) \to \bar{\Omega}^2(C'_{k+1}//C'_k)$$

maps dw_{k+2} to $w_{k+1} \otimes w_{k+1}$.

The cohomology of C'_{k+1} can now be computed inductively from $\xi(k)$. As the differentials in $\xi(k)$ commute with the action of $R' = (Z/2)[t, t^{-1}]$, we omit coefficients of the form t^n.

Lemma 3.2. *$\xi(1)$ collapses, so $H^{**}(C'_2) \cong P_{R'}[x] \otimes_{R'} P_{R'}[y]$, where*

$$x = \{w_1\} \quad \textit{and} \quad y = \{w_1 + w_2\}.$$

Proof. $H^{**}(XC'_1) = P_{R'}[x]$ and $H^{**}(C'_2//C'_1) = P_{R'}[\{w_2\}]$. According to 3.1, $d_2^{0,1}\{w_2\} = 0$; since the differentials are derivations, $E_2^{**}(1) = E_\infty^{**}(1)$. Thus

$$H^{**}(C'_2) = P_{R'}[x] \otimes_{R'} P_{R'}[\{w_2\}] = P_{R'}[x] \otimes_{R'} P_{R'}[y].$$

Lemma 3.3. (a) *For $k>1$*

$$H^{**}(C'_{k+1}) \cong P_{R'}[x] \otimes_{R'} E_{R'}(y) \otimes_{R'} P_{R'}[z_{k+1}],$$

(b) *For $k>2$, $i_*: H^{**}(C'_k) \to H^{**}(C'_{k+1})$ maps x to x, y to y and z_k to* 0.

Proof. (a) For $k>2$ suppose that it has been shown that

$$H^{**}(C'_k) \cong P_{R'}[x] \otimes_{R'} E_{R'}(y) \otimes_{R'} P_{R'}[z_k].$$

It is clear that $H^{**}(C'_{k+1}//C'_k)$ is $P_{R'}[\{w_{k+1}\}]$. According to 3.1, $d_2^{0,1}(\{w_{k+1}\}) = z_k$, while $\{w_{k+1}\otimes w_{k+1}\}\in E_2^{0,2}$ survives to z_{k+1}, since the 'fiber' edge morphism $E_\infty^{0,2}\to E_2^{0,2}$ is the map

$$F^0H^2(C'_{k+1})/F^1H^2(C'_{k+1})\to H^2(C'_{k+1}//C'_k)$$

induced by the projection

$$\bar{\Omega}^2(C'_{k+1})\to\bar{\Omega}^2(C'_{k+1}//C'_k).$$

Thus $E_3^{**}(k)\cong P_{R'}[x]\otimes_{R'}E_{R'}(y)\otimes P_{R'}[z_{k+1}]$ and $E_3^{**}(k)=E_\infty^{**}(k)$.

In case $k=2$, $H^{**}(C'_2)=P_{R'}[x]\otimes_{R'}P_{R'}[y]$ and $d_2^{0,2}(\{w_3\})=y^2$; otherwise the determination of $H^{**}(C'_3)$ is as above. This completes the proof (a). Part (b) is immediate since the 'base' edge morphism

$$E_2^{p,0}\to E_\infty^{p,0}\quad\text{is}\quad H^p(C'_k)\to H^p(C'_{k+1}).$$

The proof of 1.3 now follows from 3.3 since $C'=\operatorname{dir\,lim} C'_k$.

The next order of business is the proof of 1.2. First note that the left and right actions of R' on C' coincide since $2^{d(n)}g_n=v^n-u^n$. Now recall that the elements $p_n(u,v)=u^n\binom{w}{n}$ span C over $Z_{(2)}[u,v,u^{-1},v^{-1}]$ so that the images of the elements $\binom{w}{n}$ under the map $j:C\to C'$ span C' over $(Z/2)[t,t^{-1}]$.

Lemma 3.4. *The set* $\{j\binom{w}{2n}\mid n=0,1,2,\ldots\}$ *is a basis for* C' *over* R'.

Proof. To see that this set spans C', we note that

$$j\binom{w}{2k+1}=j\binom{w}{2k}\quad\text{for } k>0,$$

a fact which follows at once from the identity

$$w\binom{w}{2k}=2k\binom{w}{2k}+(2k+1)\binom{w}{2k+1}.$$

To show independence for the set it is sufficient to show it is independent over $Z/2$. Suppose there are numbers $a_i\in Z/2$ such that

$$\sum_{i=0}^{n}a_ij\binom{w}{2i}=0$$

in C'. This relation entails and is entailed by the relation that

$$\tfrac{1}{2}\left(\sum_{i=0}^{n}a_i\binom{w}{2i}\right)$$

is integral for some numbers $a_i\in Z_{(2)}$. We will show that this relation implies that $a_i\equiv 0 \pmod 2$; that is, $a_i=2A/(2B+1)$, $A,B\in Z$.

Now in order that

$$F(u,v)=\tfrac{1}{2}\sum_{i=0}^{n} a_i\binom{w}{2i}$$

be integral it is necessary (and sufficient) that $F(1,2r+1)$ lies in $Z_{(2)}$ for all $r\in Z$. To begin with, $F(1,1)$ must be in $Z_{(2)}$; but $F(1,1)=\tfrac{1}{2}a_0$, so $a_0=2K/(2L+1)$ with K, L in Z. Likewise $F(1,3)$ must be in $Z_{(2)}$ but

$$F(1,3)=\tfrac{1}{2}a_0+\tfrac{3}{2}a_1=K/(2L+1)+\tfrac{3}{2}a_1.$$

Thus $a_1=2M/(2N+1)$ with $M,N\in Z$. Continuing in this way, we see that $a_i\equiv 0 \pmod 2$ for each $i=1,2,\ldots,n$.

The last step in determining the *algebra* structure of C' is given in

Lemma 3.5. (a) *If* $2n=\sum_{i=1}^{m} a_i 2^i$ *with* $a_i=0,1$, *then*

$$j\binom{w}{2n}=\prod_{i=1}^{m} w_i^{a_i} \quad \textit{where } w_i=j\binom{w}{2^i}.$$

(b) *For* $k\geq 1$, $w_k^2=w_k$.

Proof. For (a) it suffices to show that

$$\tfrac{1}{2}\left(\binom{w}{2n}-\prod_{i=1}^{m}\binom{w}{2^i}^{a_i}\right)$$

is integral. However the integers r for which $\binom{r}{2n}\equiv 0 \pmod 2$ are precisely the integers r for which

$$\prod_{i=1}^{m}\binom{r}{2^i}^{a_i}\equiv 0 \pmod 2,$$

and the desired integrality follows. The proof of (b) is similar.

From the formula for Ψ in C, it is easy to show that $\Delta w=w\otimes w$. This formula, the formulas

$$\binom{w}{n}=\frac{1}{n!}\sum_{r=0}^{r} s_n^r w^r \quad\text{and}\quad w^r=\sum_{i=1}^{r} i!\, S_n^i\binom{w}{i}$$

(which define the Stirling numbers of the first and second kind) and some power series algebra over $Z/4$ entail part (c) of 1.2. The details are given in [7].

We close this section with a brief discussion of 2.2, the proof of which involves the stable Adams operations

$$\Psi^{2r+1}: K_*(K)\to K_*(K)$$

presented in [4]. It follows from its definition that Ψ^{2r+1} is a map of left-$K_*(K)$ comodules and that $\Psi^{2r+1}f(u,v)=f(u,(2r+1)v)$. Adams [3] has given the following criterion for integrality of an element $f(u,v)\in Q[u,v,u^{-1},v^{-1}]$.

Lemma 3.6. *Let $f(u,v)$ be a Laurent polynomial for which*
(a) *$\Psi^3 f-f$ is in $K_*(K)$, and*
(b) *$\varepsilon\Psi^3 f$ and $\varepsilon\Psi^{-3}f\in Z_{(2)}[t,t^{-1}]$ (where $\varepsilon: K_*(K)\to Z_{(2)}[t,t^{-1}]$ is the augmentation). Then f is integral, that is $f\in K_*(K)$.*

We indicate the proof of 3.6. Suppose given an odd integer $2r+1$ and Laurent polynomial $f=\sum a_{ij}u^i v^j$. If 2^N is the largest power of 2 occurring in the denominators of the a_{ij}, then there is an integer $k\geq 0$ so that $2r+1\equiv m \pmod{2^N}$ where $m=3^k$ or $m=-3^k$ (since 3 and -3 generate the group of units of $Z/2^N$). Then $\Psi^{2r+1}f-\Psi^m f$ is integral so that one only needs to check Condition (2.1) when $2r+1$ is 3^k or -3^k. It is easy to show that (a) and (b) of 3.6 insure that (2.1) holds for these values of $2r+1$.

To obtain 2.2 one uses the definition of $\zeta(y,a)$ and the fact that Ψ^3 is a map of left-comodules to show that

$$\Psi^3\zeta(y,a)-\zeta(y,a)\in K_*(K)$$

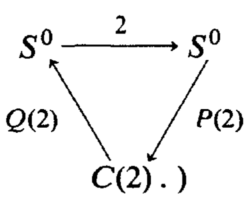
for any $a\in Q$ with $\varepsilon\Psi^3(y,a)\in Z_{(2)}[t,t^{-1}]$. The result is now immediate from 3.6.

4. Proofs of 1.4 and 1.5

The proofs are given in order. Recall that the hypotheses for 1.4 are
(a) the spectral sequence converges to $\pi_*^K(S^0)$, and
(b) $\pi_{-1}^K(S^0)=\pi_{-1}(S^0)=0$ and $\pi_2^K(S^0)=\pi_2(S^0)=Z/2$.

Lemma 4.1. (a) Tors $\pi_0^K(S^0)\cong(Z/2)\langle g_{-1}g_2\rangle$,
(b) $d_3(g_2)=g_1^2 g_2 g_{-1}$.

Proof. (a) We first show that Tors $\pi_0^K(S^0)$ is at most $Z/2$. Since $\pi_{-1}^K(S^0)=0$, none of the elements $g_1^{2k}g_{-k}$ $(k\geq 1)$ survives, a fact which entails that none of the elements $g_1^{2k+1}g_{-k}$ with $k\geq 1$ can survive. It remains to show that $g_{-1}g_2$ survives for which it suffices to show that $ty\in H^{1,2}(C')$ is the e-invariant of some $\alpha\in\pi_1^K(C(2))$. (Here $C(2)$ is the mapping cone in the cofiber triangle



.)

For given such an α, $\delta^1(ty)=g_{-1}g_2$ by Corollary 2.5 on the one hand and $Q(2)\alpha$

projects to $g_{-1}g_2$ by a geometric boundary argument on the other. To construct α, we proceed as follows. According to Adams [2], there are K-equivalences

$$A:\Sigma^6 C(2)\to\Sigma^{-2}C(2) \quad\text{and}\quad A':C(2)\to\Sigma^{-2}C(2)$$

associated to $8\sigma\in\pi_7(S^0)$ and $\eta\in\pi_1(S^0)$. Let $\bar{\eta}$ be the coextension $A'P(2):S^0\to\Sigma^{-2}C(2)$. Then we have a commutative diagram

$$\begin{array}{ccccc}
S^7 & \longrightarrow & S^0 & \longrightarrow & C(\sigma)\\
\| & & \downarrow{\scriptstyle\bar{\eta}} & & \downarrow{\scriptstyle\bar{\bar{\eta}}}\\
S^7 & \xrightarrow{\bar{\eta}\sigma} & \Sigma^{-2}C(2) & \longrightarrow & C(\bar{\eta}\sigma)
\end{array}$$

The element $\alpha\in\pi_1^K(C(2))$ is represented by the 6-fold desuspension of the diagram

$$S^7\xrightarrow{\bar{\eta}\sigma}\Sigma^{-2}C(2)\xleftarrow{A}\Sigma^6 C(2).$$

To compute $e(\alpha)$ we note that there are elements

$$a\in K_8(S^{-2}\cup_2 e^{-1}\cup_{\bar{\eta}\sigma}e^8)\quad\text{and}\quad b\in K_8(S^0\cup_\alpha e^8)$$

which map to the generator $\iota_8\in K_8(S^8)$ under the obvious projections and maps

$$\tilde{\imath}':S^{-2}\cup_2 e^{-1}\cup_{\bar{\eta}\sigma}e^8\to K\wedge\Sigma^{-2}C(2)$$

and

$$\tilde{\imath}:S^0\cup_\sigma e^8\to K$$

which extend $\Sigma^{-2}C(2)\to K\wedge\Sigma^{-2}C(2)$ and $S^0\to K$ respectively. Then $e(\bar{\eta}\sigma)$ is represented by $\tilde{\imath}'_*(a)\in K_8(K\wedge\Sigma^{-2}C(2))$ [6] and there is a commutative diagram

$$\begin{array}{ccccc}
K_8(S^{-2}\cup_2 e^{-1}\cup_{\bar{\eta}\sigma}e^8) & & \xleftarrow{\quad\bar{\bar{\eta}}_*\quad} & & K_8(S^0\cup_\sigma e^8)\\
\downarrow{\scriptstyle\tilde{\imath}'_*} & & & & \downarrow{\scriptstyle\tilde{\imath}_*}\\
K_8(K\wedge\Sigma^{-2}C(2)) & \xleftarrow{(1\wedge A')_*} & K_8(K\wedge C(2)) & \xleftarrow{(1\wedge P(2))_*} & K_8(K)\\
 & \nwarrow{\scriptstyle(1\wedge A)_*} & \uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong}\\
 & & K_8(K\wedge\Sigma^6 C(2)) & \xleftarrow{(1\wedge P(2))_*} & K_8(K\wedge S^6)
\end{array}$$

in which the bottom vertical isomorphisms are compositions of Bott maps. Since $\bar{\bar{\eta}}_*(b)=a$, $\tilde{\imath}_*(b)=g_4=(v^4-u^4)/16$ and α is the 6-fold desuspension of

$$S^7\xrightarrow{\bar{\eta}\sigma}\Sigma^{-2}C(2)\xleftarrow{A}\Sigma^6 C(2),$$

$e(\alpha)$ is represented by $t^{-3}(1\wedge P(2))_*(g_4)$ in $K_2(K\wedge C(2))$ but $t^{-3}(1\wedge P(2))_*(g_4)$ equals ty according to the proof of 2.6.

Finally the proof of (b). Since $g_{-1}g_2$ survives, $g_1^2g_{-1}g_2$ is permanent, so if d_3g_2 fails to hit $g_1^2g_{-1}g_2$, $\pi_2^K(S^0)$ would have order at least 4. This is a contradiction since $\pi_2^K(S^0)=Z/2$, generated by η^2 (which projects to g_1^2).

Corollary 4.2. *In* E_3, $d_3(g_{-1})=g_1^3g_{-3}$ *and* $d_3(g_1g_{-1})=g_1^4g_{-3}$.

The remaining assertions of 1.4 are covered in

Lemma 4.3. (a) *For* $n=4i$, $4i+1$ (*with* $n\neq 0$), g_n *survives.*

(b) *For* $n=4i+2$, $4i+3$: $d_3(g_n)=g_1^3g_{n-2}$.

Proof. (a) For $i\geq 0$, $g_{4(i+1)}$ and g_{4i+1} survive as they are the e-invariants of elements of $\pi_*(S^0)$ by results of Section 7 of [2]. According to 1.1,

$$g_{-4i}g_{4i+1}=g_2g_{-1},\quad i>0.$$

Applying d_r to this equation, we see that for $r>2$

$$d_r(g_{-4i})\cdot g_{4i+1}=d_r(g_2g_{-1})=0.$$

Thus for $r\geq 2$, $d_r(g_{-4i})=0$ when $i>0$. In like manner the fact that

$$g_{1-4i}g_{4i+1}=g_1^2$$

entails that $d_r(g_{1-4i})=0$ for $i>0$ and $r\geq 2$.

(b) Applying d_3 to the relation

$$g_{4i+3}g_{-4i-3}=g_1g_{-1}$$

one sees that

$$d_3(g_{4i+3})\cdot g_{-4i-3}=g_1^4g_{-3}$$

from which it follows that

$$d_3(g_{4i+3})=g_1^3\cdot g_{4i+1}.$$

For $n=4i+2$ with $i\neq 0$, use the relation

$$g_{4i+2}\cdot g_{-4i-1}=g_2g_{-1}.$$

We close this section with an illustration of how 1.5 can be deduced from 1.4.

Lemma 4.4. *If* $z\in E_2^{s,t}$ *with* $s>3$, *then* z *fails to survive to* $E_\infty^{s,t}$.

Proof. The proof can be broken down into cases according the mod 8 value of $t-s$. The case $(t-s)\equiv 1 \pmod 8$ is typical and we give it in detail. We need only concern ourselves with $s \equiv 2m+1$ $(m>1)$ and $t=s+8k+1$. Then $E_2^{s,t} \cong Z/2$ is generated by

$$g_2^{s-1} \cdot g_{4k+4-s}.$$

For $s \equiv 1$ (mod 4), this generator survives to $E_3^{s,t}$ and bounds, while for $s \equiv 3$ (mod 4) d_3 of it is nonzero. The cases of the other mod 8 values of $t-s$ are similar.

The proof of 1.5 is completed by treating the elements on the 3 line of the spectral sequence according to their mod 8 values of $t-s$. We continue with the illustration $(t-s) \equiv 1$ (mod 8). If $t-s=1$, $E_2^{3,4}$ is generated by $g_1 g_2 g_{-1}$, while $E_2^{3,8k+4}$ with $k \neq 0$ is generated by $g_1^2 g_{4k}$; each is a permanent cycle which cannot bound. Thus $\pi_{8k+1}^K(S^0)$ has order 4. According to Adams [2] there is an element of order 2 in $\pi_{8k+1}(S^0)$ whose e-invariant is g_{4k+1} so the extension is split.

The case when $(t-s) \equiv 3$ (mod 8) is analogous. From the knowledge of the differentials it is easy to show that $\pi_{t-s}^K(S^0)$ is an extension of $Z/4$ by $Z/2$. As with the previous case, results from $J(X)$–IV (Section 7 in [2]) resolve the extension problem.

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